THE KUNEN–MARTIN THEOREM

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A binary relation $\prec$ on a set X ($\prec$ need not be transitive) is said to be *wellfounded* if there is no infinite sequence (x_n) in X such that $x_0 \succ x_1 \succ \ldots$. Equivalently, $\prec$ is wellfounded if for any non-empty set $A \subseteq X$ there is an $x \in A$ such that $y \not\prec x$ for all $y \in A$.

In many applications, $\prec$ will indeed be a strict partial order, but, for example, the membership relation ϵ is wellfounded on any subset of the universe of sets, but not in general transitive

Now, if $\prec$ is a wellfounded relation on X, we can define a wellfounded tree $T_\prec$ on X by setting $\emptyset \in T_\prec$ and $(x_0, x_1, \ldots, x_n) \in T_\prec$ for all descending sequences $x_0 \succ x_1 \succ \ldots \succ x_n$. Using this, we can define the *rank* of the relation $\prec$ by

$$\rho(\prec) = \rho_{T_\prec}(\emptyset).$$

Exercise 1. Suppose $\prec_1$ and $\prec_2$ are wellfounded relations on sets X_1 and X_2. Assume that $\phi\colon X_1 \to X_2$ is a *homomorphism* from $\prec_1$ to $\prec_2$, i.e., that $x \prec_1 y \Rightarrow \phi(x) \prec_2 \phi(y)$. Show that $\rho(\prec_1) \leqslant \rho(\prec_2)$. [Hint: ϕ induces a mapping between the trees $T_{\prec_1}$ and $T_{\prec_2}$.]

Definition 2. Let κ be a cardinal number and A be a non-empty set. A subset $B \subseteq A^{\mathbb{N}}$ is said to be κ-Souslin if there is a tree S on $A \times \kappa$ such that $B = \mathrm{proj}_{A^{\mathbb{N}}}\big([S]\big)$.

Thus, ω-Souslin sets in Baire space $\mathbb{N}^{\mathbb{N}}$ or its finite products $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \cong (\mathbb{N} \times \mathbb{N})^{\mathbb{N}}$ are simply the analytic sets.

Theorem 3 (K. Kunen – D. A. Martin)**.** *Fix an infinite cardinal number κ and suppose $\prec$ is a wellfounded relation on $\mathbb{N}^{\mathbb{N}}$ that is κ-Souslin seen as a subset of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \cong (\mathbb{N} \times \mathbb{N})^{\mathbb{N}}$. Then $\rho(\prec) < \kappa^+$.*

Proof. Since $\prec$ is κ-Souslin, we can find a tree S on $\mathbb{N} \times \kappa \times \mathbb{N}$ such that for any $x, y \in \mathbb{N}^{\mathbb{N}}$,

$$x \succ y \;\Leftrightarrow\; \exists \alpha \in \kappa^{\mathbb{N}} \;\; (x, \alpha, y) \in [S].$$

Let W be the set of all finite strings

$$w = (s_0, u_1, s_1, u_2, \ldots, s_{n-1}, u_n, s_n),$$

where $s_i \in \mathbb{N}^n$, $u_i \in \kappa^n$ and $(s_{i-1}, u_i, s_i) \in S$ for all $1 \leqslant i \leqslant n$. Clearly, $|W| \leqslant \kappa$. Define also a wellfounded relation $\prec^*$ on W by setting

$$(s_0, u_1, s_1, u_2, \ldots, s_{n-1}, u_n, s_n) \;{}^*\!\!\succ\; (s'_0, u'_1, s'_1, u'_2, \ldots, s'_{m-1}, u'_m, s'_m),$$

if and only if $n < m$ and $s_i \subseteq s'_i$, $u_i \subseteq u'_i$ for all $i \leqslant n$. Thus, W can be seen as the set of attempts at constructing an infinite descending chain in $\prec$ along with attempts at producing witnesses for this and one has $w \;{}^*\!\!\succ\; w'$ if w' is a better attempt than w at producing an even longer descending sequence. To see that $\prec^*$ is wellfounded, just note that if $w_n = (s^n_0, u^n_1, s^n_1, \ldots, u^n_n, s^n_n)$ were $\prec^*$-decreasing, then $x_i = \bigcup_n s^n_i \in \mathbb{N}^{\mathbb{N}}$ and

$\alpha_i = \bigcup_n u_i^n \in \kappa^{\mathbb{N}}$ would be welldefined and $(x_{i-1}, \alpha_i, x_i) \in [S]$ for all i, meaning that $x_0 \succ x_1 \succ \ldots$, which is impossible. It follows that $\rho(\prec^*) < |W|^+ \leqslant \kappa^+$.

Now, for all pairs $x \succ y$, pick some $\alpha_{x,y} \in \kappa^{\mathbb{N}}$ such that $(x, \alpha_{x,y}, y) \in [S]$, and define $\phi\colon T_\prec \to W$ by

$$\phi(x_0, x_1, \ldots, x_n) = (x_0|n, \alpha_{x_0,x_1}|n, x_1|n, \alpha_{x_1,x_2}|n, \ldots, \alpha_{x_{n-1},x_n}|n, x_n|n).$$

Then ϕ is a homomorphism of $\subsetneq$ to $\ ^*\succ$, whereby $\rho(\prec) = \rho_{T_\prec}(\emptyset) < \rho(T_\prec) \leqslant \rho(\prec^*) < \kappa^+$. □

Corollary 4. *Analytic wellfounded relations on Polish spaces have rank $< \omega_1$.*

Proof. If $\prec$ is an analytic wellfounded relation on a Polish space X, by Borel embedding X into $\mathbb{N}^{\mathbb{N}}$ we can suppose that $\prec$ is an analytic, i.e., ω-Souslin, wellfounded relation on $\mathbb{N}^{\mathbb{N}}$ and thus has rank $< \omega^+ = \omega_1$. □

Corollary 5. *Suppose $A \subseteq WO$ is analytic as a subset of the Polish space* $\mathrm{Tr}_{\mathbb{N}}$. *Then*

$$\sup\{\rho(T) \bigm| T \in A\} < \omega_1.$$

Proof. Define an analytic wellfounded relation $\prec$ on $\mathrm{Tr}_{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ by

$$(T,t) \prec (S,s) \iff T = S \in A \;\&\; s \subsetneq t \in T.$$

Then $\sup\{\rho(T) \bigm| T \in A\} \leqslant \rho(\prec) < \omega_1$. □

Though we are mostly interested in analytic and coanalytic sets, there are some very interesting applications of the Kunen–Martin Theorem for sets of higher complexity, but for that we need first to investigate the extent of ω_1-Souslin sets.

Definition 6. (Kleene–Brouwer) We define the *Kleene–Brouwer ordering* $<_{KB}$ on $\mathbb{N}^{<\mathbb{N}}$ by

$$s <_{KB} t \quad \Leftrightarrow \quad t \subsetneq s \text{ or } \big[\exists i < \min\{|s|,|t|\} \quad s|_i = t|_i \;\;\&\;\; s(i) < t(i)\big].$$

Exercise 7. Show that a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is wellfounded if and only if T is wellordered under $<_{KB}$. It follows that T is wellfounded if and only if there is an order embedding of $(T, <_{KB})$ into $(\omega_1, <)$.

Theorem 8 (J. R. Shoenfield)**.** *Every Σ^1_2 set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is ω_1-Souslin.*

Proof. Assume first that A is Π^1_1. Then, since $\sim A$ is Σ^1_1, there is a tree T on $\mathbb{N} \times \mathbb{N}$ such that

$$x \notin A \Leftrightarrow \exists y \in \mathbb{N}^{\mathbb{N}} \;\; (x,y) \in [T],$$

or equivalently

$$\begin{aligned} x \in A \; &\Leftrightarrow T(x) = \{t \in \mathbb{N}^{<\mathbb{N}} \bigm| (x|_{|t|}, t) \in T\} \text{ is wellfounded} \\ &\Leftrightarrow (T(x), <_{KB}) \text{ embeds into } (\omega_1, <). \end{aligned}$$

Enumerate $\mathbb{N}^{<\mathbb{N}}$ injectively as $t_0, t_1, t_2, \ldots$ in such a way that $|t_n| \leqslant n$. We define a tree S on $\mathbb{N} \times \omega_1$ by

$$(s,u) \in S \iff \forall i,j < |s| \; \big[(s|_{|t_i|}, t_i) \in T \;\&\; (s|_{|t_j|}, t_j) \in T \to \big(t_i <_{KB} t_j \leftrightarrow u(i) < u(j)\big)\big].$$

We claim that

$$x \in A \Leftrightarrow \exists y \in \omega_1^{\mathbb{N}} \;\; (x,y) \in [S].$$

To see this, note that if $(x,y) \in [S]$, then

$$t_i \in T(x) \mapsto y(i) \in \omega_1$$

is an order embedding of $(T(x), <_{KB})$ into $(\omega_1, <)$, whereby $x \in A$. For if $t_i, t_j \in T(x)$, note that as $(x|_{i+j}, y|_{i+j}) \in S$, we have $(x|_{|t_i|}, t_i) \in T$ and $(x|_{|t_i|}, t_i) \in T$, whence

$$t_i <_{KB} t_j \leftrightarrow y(i) < y(j).$$

Conversely, if $x \in A$, then there is an order embedding of $(T(x), <_{KB})$ into $(\omega_1, <)$, say

$$t_i \in T(x) \mapsto y(i) \in \omega_1,$$

so if we let $y(i) = 0$ whenever $t_i \notin T(x)$, i.e., $(x|_{|t_i|}, t_i) \notin T$, then $(x, y) \in [S]$.

Now, if instead A is Σ^1_2, there is a Π^1_1 set $B \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that $x \in A \Leftrightarrow \exists y \in \mathbb{N}^{\mathbb{N}}\, (x, y) \in B$. Finding a tree S on $\mathbb{N} \times \mathbb{N} \times \omega_1$ such that $(x, y) \in B \Leftrightarrow \exists z \in \omega_1^{\mathbb{N}}\, (x, y, z) \in [S]$ and thus

$$x \in A \Leftrightarrow \exists (y, z) \in \mathbb{N}^{\mathbb{N}} \times \omega_1^{\mathbb{N}}\, (x, y, z) \in [S].$$

Identifying $\mathbb{N}^{\mathbb{N}} \times \omega_1^{\mathbb{N}}$ with $(\mathbb{N} \times \omega_1)^{\mathbb{N}}$ and $\mathbb{N} \times \omega_1$ with ω_1, the theorem follows. □

K. Gödel showed that under the hypothesis $V = L$, that is, in the universe of constructible sets, the continuum hypothesis holds and in, in fact, there is a Σ^1_2 wellordering of $\mathbb{R}$ or ordertype $< \omega_2$. Using the Kunen–Martin Theorem, we see that a Σ^1_2 wellordering of $\mathbb{R}$ already implies the continuum hypothesis.

Corollary 9. *If there is a Σ^1_2 wellordering of $\mathbb{R}$, then $2^{\aleph_0} = \aleph_1$.*